



12

The Transportation Problem

LEARNING OBJECTIVES :

After studying this unit you will be able to :

- Formulate a transportation problem
- Find initial basic feasible solution by various methods
- Find minimum transportation cost schedule
- Ascertain minimum transportation cost schedule
- Discuss appropriate method to make unbalanced transportation problems balanced
- Examine prohibited and preferred routes
- Formulate and solve transportation problem

12.1 INTRODUCTION

This chapter is devoted to special problems that belong to the so called transportation class. These special problems are quite important from the practical point of view. Their practical importance arises because several real situations can be described by systems of equations that fall into the transportation class. Sizeable applications of linear programming problems have been made in this field.

A typical transportation problem is concerned with selecting routes in a production distribution network among manufacturing plants and distribution warehouses or among regional distribution warehouses and local distribution outlets. The objective is to schedule transportation of products from sources to destination in such a way as to minimise the total transportation cost.

A transportation problem can be paraphrased by considering m factories which supply to n warehouses or distribution centers. The factories produce goods at level a_1, a_2, \dots, a_m and the demand are requirements of the distribution centers for these goods are b_1, b_2, \dots, b_n respectively. If the unit cost of shipping from i -th factory to warehouse j is c_{ij} what shipping pattern minimizes the transportation cost?

Let x_{ij} denote number of units transported from factory i to destination j



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$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1 \dots m \quad \dots\dots\dots(1)$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1 \dots n \quad \dots\dots\dots(2)$$

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \quad \dots\dots\dots(3)$$

$$x_{ij} \geq 0 \text{ for all } i, j$$

Equation 1 is interpreted as the sum of what leaves each factory (or origin) for the various warehouses (or destinations) is equal to what is produced at the factory, (2) implies that the sum of what arrives at each warehouse from the various origins is equal to the demand at the warehouse. The double sum of equation 3 represents the total transportation cost. The non-negative conditions (4) arise because negative values for any x_{ij} have no physical meaning.

Solution to the problem described by equations 1 to 4 is given under the condition that

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

From the physical point of view this condition means that the system of equations is in balance i.e., total production is equal to the total requirements. Equations (1) and (2) may be expanded as below.

$$X_{11} + x_{12} + \dots + x_{1n} = a_1$$

$$X_{12} + x_{22} + \dots + x_{2n} = a_2$$

$$X_{m1} + x_{m2} + \dots + x_{mn} = a_m$$

$$X_{11} + x_{12} + \dots + x_{m1} = b_1$$

$$X_{12} + x_{22} + \dots + x_{m2} = b_2$$

:
:
:

$$X_{1n} + x_{2n} + \dots + x_{mn} = b_n$$

This is a system of (m+n) equations in mn unknown; but the equations are not independent. Two important observations about the system of equations are worth noting.

- i) The co-efficient of x_{ij} 's are either 1 or 0
- ii) Any x_{ij} appears only once in the first m equations and once in the last n equations.

Our problem is to determine x_{ij} , the quantity that is to be shipped from the i-th origin to the j-th destination in such a way that the total transportation cost is minimum. The quantities of interest can be tabulated as below.



Destinations

Origins	D_1	$D_2 \dots$	$D_j \dots$	D_n	Available
O_1	c_{11}	$c_{12} \dots$	$c_{1j} \dots$	c_{1n}	a_1
O_2	c_{21}	$c_{22} \dots$	$c_{2j} \dots$	c_{2n}	a_2
:	:	:	:	:	:
O_i	c_{i1}	$c_{i2} \dots$	$c_{ij} \dots$	c_{in}	a_i
O_m	c_{m1}	$c_{m2} \dots$	$c_{mj} \dots$	c_{mn}	a_m
Required	b_1	$b_2 \dots$	$b_j \dots$	b_n	Total

The method for solving the class of problems consist of finding a basic feasible solution. If it is not optimal (an interactive procedure is used to improve it) then optimality test is applied to make it an optimal solution.

12.2 METHODS OF FINDING INITIAL SOLUTION TO TRANSPORTATION PROBLEMS

12.2.1 Northwest corner Rule: The idea is to find an initial to find an initial basic feasible solution i.e., a set of allocations that satisfied the row and column totals. This method simply consists of making allocations to each row in turn, apportioning as much as possible to its first cell and proceeding in this manner to its following cells until the row total in exhausted.

The algorithm involved under north-west corner rule consists for the following steps:

Steps:

1. Before allocation ensure that the total of availability and requirement is equal. If not then make same equal.
2. The first allocation is made in the cell occupying the upper left hand corner of the matrix. The assignment is made in such a way that either the resource availability is exhausted or the demand at the first destination is satisfied.
3. (a) If the resource availability of the row one is exhausted first, we move down the second row and first column to make another allocation which either exhausts the resource availability of row two or satisfies the remaining destination demand of column one.
 (b) If the first allocation completely satisfies the destination demand of column one, we move to column two in row one, and make a second allocation which either exhausts the remaining resource availability of row one or satisfies the destination requirement under column two.
4. The above procedure is repeated until all the row availability and column requirements are satisfied. Consider, for example, the following sample problem. This method does not use transportation costs which we shall bring in later in the other method.



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							Available
							10
							12
							4
							18
							20
Required	8	8	16	3	8	21	

Row wise allocation as above, is made below. The maximum that can be allocated.

8	2						10/2/0
	6	6					12/6/0
		4					4/0
		6	3	8	1		18/12/9/1/0
					20		20/0
$\frac{8}{0}$	$\frac{8}{6}$	$\frac{16}{10}$	$\frac{3}{0}$	$\frac{8}{0}$	$\frac{21}{21}$		
	$\frac{6}{0}$	$\frac{6}{0}$			$\frac{0}{0}$		

In cell (1, 1) 8 is allocated. This satisfies completely the requirements of column 1, but availabilities of row 1 are not completely exhausted. Therefore, we proceed to cell (1,2) in row 1 and allocate the remaining 2 units. We can start at cell (2,2) only since first column's requirements have been completely satisfied, there is nothing that we can allocate in its first cell (2,1). By the aforesaid procedure, we allocate 6 units to cell (2,2) and another 6 units to cell (2,3) exhausting completely the availabilities of row 2. This process is continued until we reach the cell (5,6).

12.2.2 The Least cost method:

- Before starting the process of allocation ensure that the total of availability and demand is equal. The least cost method starts by making the first allocation in the cell whose shipping cost (or transportation cost) per unit is lowest.
- This lowest cost cell is loaded or filled as much as possible in view of the origin capacity of its row and the destination requirements of its column.
- We move to the next lowest cost cell and make an allocation in view of the remaining capacity and requirement of its row and column. In case there is a tie for the lowest cost cell during any allocation, we can exercise our judgment and we arbitrarily choose cell for allocation.
- The above procedure is repeated till all row requirements are satisfied.

12.2.3 Vogel's Approximation Method (VAM)

The Vogel's Approximation Method (VAM) is considered to be superior to the northwest corner rule in that it usually provides an initial solution that is optimal or nearly so. Therefore, we shall also stick to it for the discussion ahead. However, the readers may like to try their hand on the following solved examples by the northwest corner rule and least-cost method for sake of practice. But we here apply VAM method.



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VAM entails the following steps :

Step 1: For each row of the transportation table identify the smallest and next smallest costs. Find the difference between the two costs and display it to the right of that row as "Difference" (Diff.). Likewise, find such a difference for each column and display it below that column. In case two cells contain the same least cost then the difference will be taken as zero.

	4	6	9	2	7	8	Avail.	Diff.
	3	5	4	8	10	0	10	2
	2	6	9	8	4	13	12/0	3
	4	4	5	9	3	6	4	2
	9	8	7	3	2	14	18	1
Req.	8	8	16	3	8	21/9	20	1
Diff.	1	1	1	1	1	6		

Step 2: From amongst these row and column differences, select the one with the largest difference. Allocate the maximum possible to the least cost cell in the selected column or row. If there occurs a tie amongst the largest differences, the choice may be made for a row or column which has least cost. In case there is a tie in cost cell also, choice may be made for a row or column by which maximum requirement is exhausted. Hatch that column or row containing this cell whose totals have been exhausted so that this column or row is ignored in further consideration.

Step 3: Recompute the column and row differences for the reduced transportation table and go to step 2. Repeat the procedure until all the column and row totals are exhausted.

The VAM is applied to the previous problem on the table above as an illustration. Entered in the table are the given unit shipping costs.

The arrow indicates that the last column having the largest difference from amongst all rows and columns is selected for allocation. Cell (2, 6) with the least cost is picked up and

	4	6	9	2	7	8	Avail	Diff
	3	5	4	8	10	0	10	2
	2	6	9	8	4	13	4/0	2 ←
	4	4	5	9	3	6	18	1
	9	8	7	3	2	14	20	1
Req	8/4	8	16	3	8	9		
Diff	2	2	2	1	1	2		



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	4	6	9	2	3	7	8		
	3	5	4	8		10	0	12	10/7 2 ←
	2	4	6	9	8	4	13		
	4	4	5	9	3	6			18 1
	9	8	7	3	2	14			20 1
Req.	4	8	16	3/0	8	9			
Diff.	0	2	2	1	1	2			

allocated 12 units. Row 2 is hatched since its total is exhausted.

This procedure is continued in the following tables until all the columns and rows are hatched.



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				3			Avail	Diff.
	4	6	9	2	7	8	7	2
	3		5	4	8	10	0	12
	2	4						
		6	9	8	4	13		
	4	4	5	9	3	6	18/9	1
	9	8	7	3	2	14	20	5
Reqd.	4	8	16		8	9/0		
Diff.	0	2	2	2	1	2		

				3			Avail	Diff.
	4	6	9	2	7	8	7	2
	3		5	4	8	10	0	12
	2	4						
		6	9	8	4	13		
	4	4	5	9	3	6	9	0
	9	8	7	3	2	8	20/12	5←
Reqd.	4	8	16		8/0			
Diff.	0	2	2		1			

				3			Avail	Diff.
	4	6	9	2	7	8	7	2
	3		5	4	8	10	0	12
	2	4						
		6	9	8	4	13		
	4	4	8	5	9	3	9/1	0
	9	8	7	3	2	8	12	1
Reqd.	4	8/0	16					
Diff.	0	2	2?					



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				3			7/3	5 ←
4	6	9	2	7	8			
						12		
3	5	4	8	10	0			
	4							
2	6	9	8	4	13			
		8						
4	4	5	9	3	6	9	1	
9	8	7	3	2	14		12	2
Req	4/0		16					
Diff.	0		2					

Since only one column is left in the table above there is no question of finding the differences and 16 units are allocated to the various cells of this column straightway as below

	4		3	3			Availability
4	6	9	2	7	8		10
						12	
3	5	4	8	10	0		12
	4						
2	6	9	8	4	13		4
		8	1				
4	4	5	9	3	6	9	18
9	8	7	12	3	2	8	20
Req.	8	8	16	3	8	21	

So many tables above have been drawn merely for exposition. All we shall see with the following examples the initial allocation by VAM can as well be obtained in just one table.

The superiority of VAM lies in the fact that, unlike the Northwest Corner rule, not only availabilities and requirements are taken into account but also due regard is paid to the unit costs. A row and column difference actually indicates the minimum unit penalty incurred by failing to make an allocation to the smallest cost cell in the row or column.

12.3 OPTIMALITY TEST:

Now we know how to obtain the initial basic feasible solution but it remains to be tested whether it is optimal or not if it is not so, how do we go about deriving the optimal solution? Before we taken up this matter it is necessary to explain the concept of **independence and non-independence** amongst allocations. Towards this, consider the following transportation tables of different problems. The allocations in these are shown by plus signs.



	+			+	
	+			+	
					+

+				+	
+			+		
			+		+

		+			
				+	+
+		+			
+		+		+	
					+
		+			

The lines constitute, in these allocation pattern what is known as loops. A loop may or may not involve all the allocations. It consists of (at least 4) horizontal and vertical lines with an allocation at each corner which, in turn is a join of a horizontal and vertical line. At this stage the last loop above is to be particularly noted. Here, two lines intersect each other at cell (2,4) and do not simply join; therefore, this is not to be regarded as a corner. Such allocations in which a loop can be formed are known as non-independent, whereas those in which a loop cannot be formed as regarded as independent. Towards explanation of this nomenclature consider the following allocation pattern with a loop in it.

4		6		10
8		2		10
			5	5
12		8	5	

It is possible to progressively adjust the allocations along the corners of the loop without violating the row and column totals. One adjusted allocations pattern for the above table is show below as an example.

$4-2=2$		$6+2=8$		
$8+2=10$		$2-2=0$		
			5	



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But such a possibility of reallocation does not exist where a loop cannot be formed i.e., allocations are independent.

Coming back now to the optimality test to be described shortly it can be applied to a transportation table if it satisfies the following conditions.

(1) it contains exactly $m+n-1$ allocations where m and n represent the number of rows and column of the table.

(2) These allocations are independent i.e. a loop can not be performed by them

Without giving the proof, we may also mention that whenever there are $m+n-1$ allocation and a tick is placed in an empty cell one and only one loop can be passed through the ticked cell and some are all of the $m+n-1$ allocation.

	4		3	3			Availability
4	6	9	2	7	8		10
3	5	4	8	10	0	12	12
2	6	9	8	4	13		4
4	4	8	1	9	3	6	18
9	8	7	12	3	2	8	20
Req.	8	8	16	3	8	21	

Cell Evaluations:

Consider the initial solution to the transportation problem on hand as obtained by VAM and reproduced above.

The allocations are $m+n-1$ in number and independent. It may be stated in passing that initial allocations obtained by Northwest Corner rule or VAM are always in independent positions though they may be $m+n-1$ or less than $m+n-1$ in number. The motivation in the optimality test is to see if it is possible to improve upon the existing solution. In the simplex language, are any of the net evaluations negative? Before we attempt this for the problem on hand let us see what net evaluations mean in the context of transportation problems. As in the simplex method, we want to replace a basic variable with a non-basic variable i.e., want to zeroise an existing allocation cell and instead make some allocation in an empty cell. The problem then boils down to determination of an “outgoing allocation” and an “incoming allocation” that brings us closest to the optimal, if at all possible.



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	4		✓	3	3		Availability
4	6	9	2	7	8		10
3	5	4	8	10	0	12	12
2	4	9	8	4	13		4
4	4	8	1			9	18
9	8	7	12	3	2	8	20
Req.	8	8	16	3	8	21	

Towards this, we arbitrarily select the empty cell (1, 2), put a tick in it meaning that we want to make some allocation in it and zeroise one (or more) existing allocating. For reason mentioned above, one and only one loop can be formed in the allocation pattern. It is shown in the table above. If we bring one unit to the ticket cell, other allocations on the corners of the loop would be adjusted as below.



This would mean lowering of the cost by

$$6 \times 1 - 9 \times 1 + 5 \times 1 - 4 \times 1 = 2$$

This then is the net evaluation of the ticketed cell and obviously it would pay to bring this cell into solution since the cost would be reduced by 2 per unit transferred to this along the loop. Likewise, we can compute cell evaluations for all the empty cells and select that as the incoming one which has the most -ve cell evaluation.

The procedure would consist of ticking each empty cell, forming a loop involving the tick and computing the cell evaluation for a unit transfer to the ticket cell along the loop. Obviously this would be a rather lengthy procedure. Fortunately there exists a much easier way of computing cell evaluations for all empty cells at one stroke. This is what we described as the optimality test above and it is applicable to transportation tables containing exactly $m+n-1$ independent allocations. It consists of the following steps:

1. Determine a set of $m+n$ numbers

$$u_i; i = 1, 2, \dots, m$$

$$v_j; j = 1, 2, \dots, n$$



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such that, for each occupied cell

$$c_{ij} = u_i + v_j$$

2. Compute the cell evaluation Δ_{ij} for each empty cell (i, j) by the relationship

$$\Delta_{ij} = c_{ij} - (u_i + v_j)$$

Let us apply this optimality test to the problem on hand. Below is the table indicating just the unit costs in allocated cells in the problem on hand. We set $u_1 = 0$ for row 1. it could be assigned any other value but zero has been assigned for ease in subsequent computations.

Also it is desirable to select that row or column which contains maximum number of allocations. In this case row 1 has a maximum of 3 allocations. Now, we wish to find the value of v_1 , v_3 and v_4 for the column of these 3 allocations which can be accomplished as below.

$$U_1 + v_1 = 4 \text{ Since } u_1 = 0 \text{ therefore, } v_1 = 4$$

$$U_1 + v_3 = 9 \quad v_3 = 9$$

$$U_1 + v_4 = 2 \quad v_4 = 2$$

						u_1
	4		9	2		0 (u_1)
					0	-10 (u_2)
	2					-2 (u_3)
		4	5		6	-4 (u_4)
			7		2	-2 (u_5)
V_j	4(v_1)	8(v_2)	9(v_3)	2(v_4)	4(v_5)	10(v_6)

Values of $v_1 = 4$, $v_3 = 9$, $v_4 = 2$ have been entered at the foot of each volume. Now let us proceed with v_1 , v_3 and v_4 in turn.

$$v_1: \quad u_3 + v_1 = 2 \quad \text{since } v_1 = 4, \text{ therefore, } u_3 = -2 \text{ is entered}$$

$$v_3: \quad u_4 + v_3 = 5 \quad \text{since } v_3 = 9, \text{ therefore, } u_4 = -4 \text{ is entered}$$

v_4 : No more allocation in its column and we ignore it.

Having established and entered values of u_2 and u_4 let us take up these in turn.

u_2 : No more allocation in its row and we ignore it.

$$u_4: \quad u_4 + v_2 = 4 \quad \text{since } u_4 = -4 \quad \text{Therefore, } v_2 = 8 \text{ is entered}$$

$$u_4 + v_6 = 6 \quad \text{since } u_4 = -4 \quad \text{therefore, } v_6 = 10 \text{ is entered}$$

Proceeding in this manner we fill up all u_i 's and v_j 's

In the following table, we derive $u_i + v_j$ for each empty cell.



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		8			4	10
-6	}	-2	}	-1	-8	
		6		7	0	2
0				-2	0	
2		6		0		8

$$u_i + v_j$$

The following table gives the cell evaluations derived by subtracting above figures from the original unit costs.

	-2			3	-2
9	7	5	16	16	
	0	2	8	2	5
4			11	3	
7	2		9		6

$$\Delta_{ij} = C_{ij} - (u_i + v_j)$$

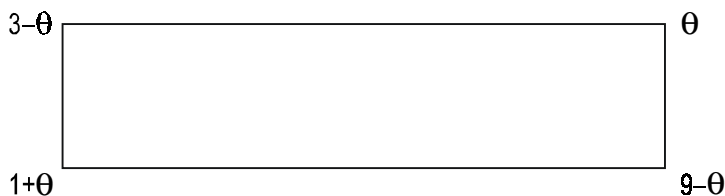
Since some of the Δ_{ij} s are -ve the initial solution is not optimal. It would pay the most to zeroise an existing allocation and allocate as much as possible to the most -ve Δ_{ij} cell (6, 1). There are two cells having the same most -ve Δ_{ij} . We pick up one of them arbitrarily for ticking.

This is done below is the allocation table by ticking cell (1,6) and involving it in the only loop.

4		3	3		
					12
4					
	8	1			9
		12		8	

Transferring the maximum amount to the ticked cell

Reallocation is done by transferring the maximum amount to the ticketed cell. The rule for obtaining the maximum amount (say, θ_{\max}) that can be transferred to the ticked cell is derived below:





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Starting with the ticked cell θ is added to and subtracted from the corner allocations alternately.

$$\theta_{\max} = \min \text{ of } \begin{array}{l} 3 - \theta = 0 \\ 9 - \theta = 0 \end{array} = 3 \text{ Thus } \theta_{\max} = 3$$

Reallocation along this loop is shown below for clarity:

Complete reallocations follow:

4			3		3
					12
4					
	8	4	4		6

Reallocation matrix
(optimal as tested below)

The optimality test is performed below:

4			2		8	u_j
					0	0
2					0	-8
	4	5			6	-2
		7		2		-2
4	6	7	2	2	8	0

	6	7		2	
-4	-2	-1	-6	-6	
	4	5	0	0	6
2			0	0	
4	6		2		8

$(u_j + v_j)$ matrix



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	0	2		5	
7	7	5	14	16	
	2	4	8	4	7
2			9	3	
5	2		1		6

δ_{ij} matrix

Since none of the Δ_{ij} is -ve this solution is optimal.

Least cost	4x4	=	16	
	3x2	=	6	
	3x8	=	24	
	0x12	=	0	
	4x2	=	8	
	8x4	=	32	
	4x5	=	20	
	6x6	=	36	
	12x7	=	84	
	8x2	=	<u>16</u>	
			<u>242</u>	Answer

12.4 SPECIAL CASES

12.4.1 Unbalanced transportation problem

Illustration

A company has factories at A, B and C which supply warehouse at D, E, F and G. Monthly factory capacities are 160, 150 and 190 units respectively. Monthly warehouse requirements are 80, 90, 110 and 160 units respectively. Units shipping costs (in Rs.) are as follows:

		To			
		D	E	F	G
From	A	42	48	38	37
	B	40	49	52	51
	C	39	38	40	43

Solution

Availabilities	= 160 + 150 + 190 = 500
Requirements	= 80 + 90 + 110 + 160 = 440
Availabilities - Requirement	= 500 - 440 = 60



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Therefore, a dummy warehouse H is introduced, and initial solution is obtained below by VAM in just one table.

	D	E	F	G	H	Available	Diff.
A	42	48	38	37	0	160/0	37/1/1/1
B	40	49	52	51	0	150/90/10/0	48*/9/11*/1
C	39	38	40	43	0	190/100/0	38/1/1/3
Reg.	80/0	90/0	110/10/0	160/0	60/0		
Diff.	0	0	0	0	0		
	1	10*	2	6*	0		

12.4.2 Degeneracy : We know that a transportation problem has $m + n - 1$ basic variables which means that the number of occupied cells in such a solution is one less than the number of rows plus the number of columns. It may happen sometimes that the number of occupied cells is less than $m + n - 1$. Such a solution is called a degenerate solution. We handle such a situation by introducing an infinitesimally small allocation e in the least cost and independent cell.

In this example, since there are only 6 (one less than $m+n - 1$) allocations, an infinitesimally small allocation e is placed in the least cost and independent cell (1, 5). This solution is tested for optimality below. (N.B.: if allocations were $m + n - 2$ we would place two e 's, e_1, e_2 which are virtually zero in the 2 least cost independent cells). This device enables us to apply to optimality test ($m + n - 1$) allocations.

					V_j
			37	0	0
40		52		0	0
	38	40			-12
V_i	40	50	52	37	0

40	50	52			$(u_i + v_j)$ matrix
	50		37		
28			25	-12	

2	-2	-14			Δ_{ij} matrix
	-1		14		
11			18	12	



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Since there are $-ve \ddot{A}_{ij}$'s the initial solution is not optimal. Reallocation is done below by ticking the most $-ve \ddot{A}_{ij}$ cell (1, 3) and involving it in the loop.

θ_{\max} min min			√	160	e	Note that the maximum that can be transferred to the ticked cell is e. Since e is infinitesimally small it leaves other corner allocations unaffected. (Intermediate i.e. non corner allocations are never altered in the process of reallocations).
	80		10		60	
		90	100			

		e	160		Reallocation
80		10		60	
	90	100			

This solution is tested for optimally below :

		38	37		38
40		52		0	52
	38	40			40
-12	-2	0	-1	-52	

26	36			-14	(u _i +v _j) matrix)
	50		51		
28			39	-12	

16	12			14	j matrix
	-1		0		
11			4	12	

Since there are $-ve a_{ij}$, this solution too is not optimal. Reallocation is done below :

$$\theta_{\max} = \min \left(\begin{matrix} 10 - \theta = 0 \\ 90 - \theta = 0 \end{matrix} \right)$$

			160	
80	√	10-θ		60
	90-?	100+θ		

		e	160		Reallocation
80	10			60	
	80	110			



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Since there are $-ve \Delta_{ij}$ this solution too is not optimal. Reallocation is done below.

This solution is tested for optimality below:

					u_i
		38	37		-13
40	49			0	0
0	-38	40			-11
V_j	40	49	51	50	0

27	26			-13	$(v_i + v_j)$
		51	50		
29			39	-11	

15	12			13	Δ_{ij} matrix
		1	1		
10			4	11	

Since all Δ_{ij} 's are +ve, this solution is optimal. The student may compute the minimum total cost.

12.4.3 Multiple solution case

Illustration

Solve the following transportation problem

	D1	D2	D3	D4	
Q1	5	3	6	2	19
Q2	4	7	9	1	37
Q3	3	4	7	5	34
	16	18	31	25	

Solution

The initial solution is obtained below by Vogel's method

	D1	D2	D3	D4	Available	Cost Difference
Q1	5	3	18	1	19/1/0	1/2/3
Q2	4	12	7	9	37/12/0	3*13*
Q3	3	4	7	30	34/30/0	1/1/4
Req.	16	18	31	25		
	4	0	30	0		
	0		0			
Cost	1	1	1	1		
Diff	2*	1	1			



The Transportation Problem 12.19

Since there are 6 ($= m + n - 1$) allocations the optimality test is performed below straightway:

		3	6	
	4			1
	3		7	
V_j	2	3	6	-1

u_i	0	2			-1
	2		5	8	
	1		4		0

(u_i + v_j) Matrix

	D1	D2	D3	D4
Q ₁	3			3
Q ₂		2	1	
Q ₃		0		5

Δ_{ij} matrix

Since all Δ_{is} 's are positive the initial solution itself is optimal. Also, since one Δ_{ij} is zero there exists other optimal solution, too i.e. A zero Δ_{ij} for an unused square means that if this route were brought into the solution, the shipping assignments would change, yet the total transportation cost would be same. To determine what this alternative optimal solution is, we follow the same procedure used for bringing any route into the solution.

12.5 MAXIMIZATION TRANSPORTATION PROBLEMS

Illustration

A company has 3 factories manufacturing the same product and 5 sole agencies in different parts of the country. Production costs differ from factory to factory, and the sales prices from agency to agency. The shipping cost per unit product from each factory to any agency is known. Given the following data, find the production and distribution schedules most profitable to the company

Production cost/unit	Max capacity No. of units	Factory i
20	150	1
22	200	2
18	125	3

1	1	5	9	4
9	7	8	3	6
4	5	3	2	7
1	2	3	4	5
80	100	75	45	125
30	32	31	34	29

} Shipping Costs

Agency j

Demand to be met

Sales price



12.20 Advanced Management Accounting

Solution

The profit matrix is derived below from the equation;

Profit = Sales – production cost – shipping cost.

		1	2	3	4	5
profit matrix	1	9	11	6	5	5
	2	-1	3	1	9	1
	3	8	9	10	14	4

The above profit matrix is converted into a loss matrix so that minimization transportation algorithms can be applied.

This is done by subtracting each entry of the above table from 14 (the largest number in the table). Subsequently we get the following loss table.

		1	2	3	4	5
1	5	3	8	9	9	
2	15	11	13	5	13	
3	6	5	4	0	10	

Since the problem is unbalanced, a dummy warehouse is added and the initial solution is obtained by VAM below:

	1	2	3	4	5	Dummy	Availability	Diff
1	50	100					150/50/0	3/2/2/2*/4
2				45	105	50	200/150/105/0	5*/6*/2/2/2
3			75		20		125/50/20/0	0/4/1/1/4
Req.	80	100	75	45	125	50		
	30	0	0	0	105	0		
	0				0			
Diff.	1	2	4	5*	1	0		
	9*				3			

Since there are 8 (= m + n - 1) allocations the optimality test is straightway performed below

	5	3					u _i
				5	13	0	-4
	6		4		10		0
v _j	9	7	7	5	13	0	-3



The Transportation Problem 12.21

		3	1	9	-4
9	7	7			
	4		2		-3

(u_i + v_j)

		5	8	0	4
6	4	6			
	1		-2		3

Δ_{ij}

$$\max = \min \begin{cases} 20 - \theta = 0 \\ 20 - \theta = 0 \\ = 20 \end{cases}$$

50	100				
			45-θ	105+?	50
30		75	√	20-?	

50	100				
			25	125	50
30		75	20		

Reallocation Matrix

5	3				
			5	13	0
6		4	0		

u_i
-6
0
-5

v _j	11	9	9	5	13	0
----------------	----	---	---	---	----	---

		3	-1	7	-6
11	9	9			
	4			8	-5

Δ_{ij} matrix

(u_i + v_j) matrix

		5	10	2	6
4	2	4			
	1			2	5

Δ_{ij} matrix

Since all Δ_{ij} are positive this solution is optimal



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50×9	=	450
100×11	=	1,100
25×9	=	225
125×1	=	125
30×8	=	240
75×10	=	750
20×14	=	<u>280</u>

Rs. 3,170 Answer

12.6 PROHIBITED ROUTES:

Sometimes in a given transportation problem, some routes may not be available. There could be several reasons for this such as bad road conditions or strike etc. In such situations, there is a restriction on the route available for transportation. To handle such type of a situation, a very large cost (or a negative profit for the maximization problem) represented by ∞ or M is assigned to each of such routes which are not available. Due to assignment of very large cost, such routes would automatically be eliminated in the final solution. The problem is solved in its usual way.

Illustration

Solve the following transportation problem:

	1	2	3	4	5	6	Stock Available
1	7	5	7	7	5	3	60
2	9	11	6	11	–	5	20
3	11	10	6	2	2	8	90
4	9	10	9	6	9	12	50
DEMAND	60	20	40	20	40	40	

Note: It is not possible to transport any quantity from factory 2 to godown 5. State whether the solution derived by you is unique.

Solution

The initial solution is found by VAM below:

Factory	Godowns						Availability	Diff.		
	1	2	3	4	5	6				
1	7	20	5	7	7	5	40	3	60/40/0	2/4/0
2	10	9	11	10	6	11	?	5	20/10/0	1/3
3	11	10	30	6	20	2	40	2	90/70/30/0	0/4/2/5
4	50	9	10	9	6	9	12	50/0	3/0	
Demand	60	20	40	20	40	40				
	50	0	10	0	0	0				
	0		0							
Diff.	2	5	0/1	4	3	2				



The Transportation Problem 12.23

The above initial solution is tested for optimality. Since there are only 8 allocations and we require $9(m+n-1)=9$ allocations, we put a small quantity in the least cost independent cell (2, 6) and apply the optimality test. Let $u_3 = 0$ and then we calculate remaining u_i and v_j

Factory	Godowns						u_j
	1	2	3	4	5	6	
		20				40	
1	7	5	7	7	5	3	-2
2	10		10			e	0
	9	11	6	11		5	0
3			30	20	40		0
	11	10	6	2	2	8	0
4	50						0
	9	10	9	6	9	12	
v_j	9	7	6	2	2	5	

Now we calculate $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for non basic cells which are given in the table below:

0		3	7	5	
	4		9	∞	
2	3				3
	3	3	4	7	7

Δ_{ij} matrix

Since all Δ_{ij} are positive, the initial solution found by VAM is an optimal solution. The final allocations are given below:

Factory	to	Godown	Unit	Cost	Value
1		2	20	5	100
1		6	40	3	120
2		1	10	9	90
2		3	10	6	60
3		3	30	6	180
3		4	20	2	40
3		5	40	2	80
4		1	50	9	<u>450</u>

Total cost Rs. 1,120

The above solution is not unique because the opportunity cost of cell (1,2) is zero. Hence alternative solution exists. Students may find that the alternative solution is as given below:



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Factory	to	Godown	Unit	Cost	Value
1		1	10	7	70
1		2	20	5	100
1		6	30	3	90
2		3	10	6	60
2		6	10	5	50
3		3	30	6	180
3		5	40	2	80
3		4	20	2	40
4		1	50	9	<u>450</u>
Total cost (Rs.)					<u>1,120</u>

Illustration

The Link manufacturing company has several plants, three of which manufacture two principal products, standard card table and deluxe card table. A new deluxe card table will be introduced which must be considered in term of selling price & costs. The selling prices are: Standard Rs. 14.95, Deluxe Rs. 18.95, and new Deluxe Rs. 21.95.

Requirements		Variable costs			Available plant capacity	
Model	Quantity	Plant A	Plant B	Plant C	Plant	Capacity
Standard	450	8.00	7.95	8.10	A	800
Deluxe	1,050	8.50	8.60	8.45	B	600
New Deluxe	600	9.25	9.20	9.30	C	700

Solve this problem by the transportation technique for the greatest contribution.

Solution

(The student should be in a position to follow the computational steps below without explanation)

6.95	7.00	6.85
10.45	10.35	10.50
12.70	12.75	12.65

Note: $6.95 = 14.95 - 8.00$

$10.45 = 18.95 - 8.50$

Contribution matrix

The following relative loss matrix has been obtained by subtracting each figure in the above contribution matrix from the highest figure of 12.75 and multiplying by 100 to get whole numbers for case in subsequent arithmetic.



The Transportation Problem 12.25

	450					
	580	575	590	Requirement	45/0	Diff.
	350		700			
	230	240	225	1050/350/0		5/5
	e	600				
	5	0	10	60/0		5
Availability	800	600	700			
	450	0	0			
	0					
Diff.	255	240	215			
	350		365			

The above allocation has been obtained by Vogel's rule.

Since there are 4 i.e., less than $m + n - 1$ allocations, a small allocation e is placed in the least cost independent cell (3, 1). The optimality test is performed below:

u_j

	580				
	580	575	575		
	230		225		
	5	0			
v_i	0	-5	-5		

($u_i + v_j$) matrix

		0	15	
		15		
			8	

Δ_{ij} matrix

Since no Δ_{ij} is $-ve$ this initial solution itself is optimal. Also since there is one zero Δ_{ij} there exists alternate optimal solution too. The largest contribution is given by.

$$6.95 \times 450 + 10.45 \times 350 + 12.75 \times 600 + 10.50 \times 700 = 21,785 \text{ answer.}$$

Note: in the examples solved herein we see that sometimes we get allocation less than $m + n - 1$ in the initial solution. We place e 's in the least cost independent cells. It is also possible that we get less than $m + n - 1$ allocations in an intermediate reallocation in which case we would place e 's in the most $-ve$ Δ_{ij} independent cell.

12.7 MISCELLANEOUS ILLUSTRATIONS

Illustration

A company wishes to determine an investment strategy for each of the next four years. Five investment types have been selected, investment capital has been allocated for each of the coming four years, and maximum investment levels have been established for each investment type. An assumption is that amounts invested in any year will remain invested until the end of the planning horizon of four years. The following table summarizes the data for this problem. The values in the body of the table represent net return on investment of one rupee up to the end of the planning horizon. For example, a rupee



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invested in investment type B at the beginning of year 1 will grow to Rs. 1.90 by the end of the fourth year, yielding a net return of Rs. 0.90.

Investment made at the beginning of year	Investment Type					Rs. Available (in '000)
	A	B	C	D	E	
	Net	Return	Data			
1	0.80	0.90	0.60	0.75	1.00	500
2	0.55	0.65	0.40	0.60	0.50	600
3	0.30	0.25	0.30	0.50	0.20	750
4	0.15	0.12	0.25	0.35	0.10	800
Maximum Rupee investment (in '000)	750	600	500	800	1,000	

The objective of this problem is to determine the amount to be invested at the beginning of each year in an investment type so as to maximize the net rupee return for the four-year period. Solve the above transportation problem and get an optimal solution. Also calculate the net return on investment for the planning horizon for four-year period.

Solution

We note that this transportation problem is an unbalanced one and it is a maximization problem. As a first step, we will balance this transportation problem.

Step 1:

Years	Investment type					Available Rs.(in '000)
	A	B	C	D	E	
	NET RETURN DATA					
1	0.80	0.90	0.60	0.75	1.00	500
2	0.55	0.65	0.40	0.60	0.50	600
3	0.30	0.25	0.30	0.50	0.20	750
4	0.15	0.12	0.25	0.35	0.10	800
Dummy	0	0	0	0	0	1000
Maximum Rs. Investment (in '000)	750	600	500	800	1,000	3,650



The Transportation Problem 12.27

Step 2: We shall now convert the above transportation problem (a profit matrix) into a loss matrix by subtracting all the elements from the highest value in the table viz. 1.00.

Investment type

Years	A	B	C	D	E	Available Rs.(in '000)
	Net Loss Data (Rs.)					
1	0.20	0.10	0.40	0.25	.0	500
2	0.45	0.35	0.60	0.40	0.50	600
3	0.70	0.75	0.70	0.50	0.80	750
4	0.85	0.88	0.75	0.65	0.90	800
Dummy	1.00	1.00	1.00	1.00	1.00	1,000
Maximum Rs. Investment (in '000)	750	600	500	800	1,000	3,650

For convenience, let us express the net loss data in the body of the above table in paise. Thereafter, we shall apply VAM to get an initial solution.

Investment type							Rs. available (000(ys)	Difference	
Year	A	B	C	D	E				
1	20	10	40	25	500	0	500/0	10	
2	45	600	35	60	40	50	600/0	5 5	
3	70	75	70	750	50		7500	20 20 20	
4	250	85	88	500	50	65	90	800/750/250/0	10 10 10 10 10
Dummy	500	100	100	100	100	500	100	1000/500/0	0 0 0 0 0
Maximum rupee investment in (000's)	750/500/0	600/0	500/0	800/50/0	1000/500/0				

Differences	25	25	20	15	50
	25	40	10	10	30
	15	–	5	15	10
	15	–	25	35	10
	15	–	25	–	10



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The initial solution got above by VAM is given below

Year	A	B	C	D	E
1	20	e	40	25	500
2	45	600	60	40	50
3	70	75	70	750	80
4	250	88	500	50	90
Dummy	500	100	100	100	500
	100	100	100	100	100

We will now test the above solution for optimality.

Total number of allocations should be $m + n - 1 = 9$ but there are only 8 allocations in the above solution which are one less than 9, hence the initial solution found above is degenerate. We introduce a small quantity e in an independent least cost cell which is (1, B) in this case to make the total number of allocation equal to 9. We introduce u_i 's and v_j 's such that $\Delta_{ij} = C_{ij} - (u_i + v_j)$ (for $i, j = 1, -5$). We assume that $u_4 = 0$ and various u_i 's and v_j 's and Δ_{ij} 's are calculated as below. Values of various Δ_{ij} 's are also written in the left hand corner of the box for non basic cells.

Year	A	B	C	D	E	
1	20	e+ θ	50	45	500+ θ	$u_1=85$
	20	10	40	2 25	0	
2	20	600	35	25	25	$u_2=60$
	45	35	60	4 40	50	
3	0	-5	10	750	10	$u_3=15$
	70	75	70	5 50	80	
4	250	-7	500	50	5	$u_4=0$
	85	88	75	6 65	90	
Dummy	500	10	10	20	500- θ	$u_5=15$
	100	100	100	100	100	
	$v_1 = 85$	$v_2 = 95$	$v_3 = 75$	$v_4 = 65$	$v_5 = 85$	

Since some of the Δ_{ij} 's are negative, the above initial solution is not optimal. Introducing in the cell (Dummy, B) with most negative Δ_{ij} an assignment e . The value of e and the new solution as obtained from above is shown below. The values of u_i 's, v_j 's are also calculated. The solution satisfies the conditions of optimality. The condition $\Delta_{ij} = C_{ij} - (u_i + v_j) \geq 0$ for non allocated cells is also fulfilled.



The Transportation Problem 12.29

Year	Investment type					
	A	B	C	D	E	
1	20	10	50	45	500	$u_1=85$
2	20	600	35	25	10	$u_2=50$
3	0	5	10	750	10	$u_3=15$
4	250	3	500	50	5	$u_4=0$
Dummy	500	e	10	20	500	$u_5=15$
	$v_1 = 85$	$v_2 = 85$	$v_3 = 75$	$v_4 = 65$	$v_5 = 85$	

Since all Δ_{ij} 's are positive, hence, the second solution obtained above is optimal. The allocation is given below.

In the year	Investment type	Amount (in '000)
1	E	500
2	B	600
3	D	750
4	A	250
4	C	500
4	D	50

The net return on investment for planning horizon of four years period is given by:
 $500 \times 1.0 + 600 \times 0.65 + 750 \times 0.50 + 250 \times 0.15 + 500 \times 0.25 + 50 \times 0.35 = \text{Rs. } 1,445 \text{ thousands}$

Illustration

XYZ and Co. has provided the following data seeking your advice on optimum investment strategy:

Investment made at the beginning of year	Net return data (in paise) of selected investments				Amount available (lakhs)
	P	Q	R	S	
1	95	80	70	60	70
2	75	65	60	50	40
3	70	45	50	40	90
4	60	40	40	30	30
Maximum Investment (Lakhs)	40	50	60	60	—



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The following additional information are also provided:

- P, Q, R and S represents the selected investments.
- The company has decided to have four years investment plan.
- The policy of the company is that amount invested in any year will remain so until the end of the fourth year.
- The values (Paise) in the table represent net returns on investment of one Rupee till the end of the planning horizon (for example, a Rupee invested in investment P at the beginning of year 1 will grow to Rs. 1.95 by the end of the fourth year, yielding a return of 95 paise). Using the above, determine the optimum investment strategy.

Solution

The given problem is an unbalanced transportation problem, which is converted into a balanced one by adding a dummy investment as given below:

Year	Net return data (in paise) of investments					Amount available (lakhs)
	P	Q	R	S	Dummy	
1	95	80	70	60	0	70
2	75	65	60	50	0	40
3	70	45	50	40	0	90
4	60	40	40	30	0	30
Maximum Investment	40	50	60	60	20	—

The values in the table represent net return on investment of one rupee till the end of the fourth year. The objective of the company is to maximize the net return. For achieving this objective, let us convert this maximization problem into minimization problem by subtracting all the elements of the above payoff matrix from the highest payoff i.e. 95, and apply Vogel's approximation method for finding the initial feasible solution.

Loss Matrix Investment type

Year	P	Q	R	S	Dummy	Amount available	Diff
1	40 0	30 15	25	35	95	70/300	15/10....
2	20	20 30	20 35	45	95	40/20/0	10/5/5/10
3	25	50	40 45	50 55	95	90/50/0	20/5/5/10/40
4	35	55	55	10 65	20 95	30/20/0	20/0/0/10/30
Maximum	40/0	50/20/0	60/40/0	60/10/0	20/0		
Difference	20	15	10	10	0		
	-	15	10	10	0		
	-	20	10	10	0		
	-	-	10	10	0		



The Transportation Problem 12.31

The solution obtained by VAM is as given below:

Loss Matrix Investment type						
Year	P	Q	R	S	Dummy	Amount available
1	40	30				
	0	15	25	35	95	70
2		20	20			
	20	30	35	45	95	40
3			40	50		
	25	50	45	55	95	90
4				10	20	
	35	55	55	65	95	30
Maximum investment	40	50	60	60	20	

This initial solution is tested for optimality. There are 8 (=m +n-1) independent allocations. Let us introduce $u_i, v_j, i = (1,2,3,4); j = (1, 2, 3, 4, 5)$ such that $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for allocated cells. We assume $u_1 = 0$ and remaining u_i 's, v_j 's and Δ_{ij} 's are calculated below.

On calculating Δ_{ij} 's for non-allocated cells, we found that their values are positive, hence the initial solution obtained above is optimal. The optimal allocations are given below:

Year	Invest in	Net return
1	Invest Rs. 40 lakhs in investment P	$0.95 \times \text{Rs. 40 lakhs} = \text{Rs. 38,00,000}$
	Rs. 30 lakhs in investment Q	$0.80 \times \text{Rs. 30 lakhs} = \text{Rs. 24,00,000}$
2	Invest Rs. 20 lakhs in investment Q	$0.65 \times \text{Rs. 20 lakhs} = \text{Rs. 13,00,000}$
	Rs. 20 lakhs in investment R	$0.60 \times \text{Rs. 20 lakhs} = \text{Rs. 12,00,000}$
3.	Invest Rs. 40 lakhs in investment R	$0.50 \times \text{Rs. 40 lakhs} = \text{Rs. 20,00,000}$
	Rs. 50 lakhs in investment S	$0.40 \times \text{Rs. 50 lakhs} = \text{Rs. 20,00,000}$
4	Invest Rs. 10 lakhs in investment S	$0.30 \times \text{Rs. 10 lakhs} = \underline{\text{Rs. 3,00,000}}$
	Total	<u>Rs. 1,30,00,000</u>

Since some of the Δ_{ij} 's = 0 the solution is not unique. Alternate solution exist.

Illustration

A particular product is manufactured in factories A, B, C, and D; and is sold at Centres 1, 2 and 3. The cost in Rs. Of product per unit and capacity in kgms per unit time of each plant is given below:



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<i>Factory</i>	<i>Cost (Rs.) per unit</i>	<i>Capacity (kgms) per unit</i>
A	12	100
B	15	20
C	11	60
D	13	80

The sales price in Rs. Per unit and the demand in kgms per unit time are as follows:

<i>Sales Centre</i>	<i>Sales price (Rs.) per unit</i>	<i>Demand (kgms) per unit</i>
1	15	120
2	14	140
3	16	60

Find the optimal sales distribution.

Solution

The given problem is a transportation problem. The profit matrix for various factories and sales counters is calculated below:

Factory	Sales Centres			Capacity (kgms per unit)
	1	2	3	
A	3	2	4	100
B	0	-1	1	20
C	4	3	5	60
D	2	1	3	80
Demand (kgms)	120	140	60	

Since this is an unbalanced transportation problem (demand > capacity), let us introduce a dummy factory with profit as Rs. 0 per unit for various sales Centres and capacity equal to sixty units. The resulting matrix would be as below:

Factory	Sales Centres			Capacity (kgms per unit)
	1	2	3	
A	3	2	4	100
B	0	-1	1	20
C	4	3	5	60
D	2	1	3	80
Dummy	0	0	0	60
Demand (kgms)	120	140	60	



The Transportation Problem 12.33

The above profit matrix can be converted into a loss matrix by subtracting all its elements from the highest payoff of the matrix i.e. 5. The loss matrix so obtained is given below.

Factory	Sales Centres			Capacity (kgms per unit)
	1	2	3	
A	2	3	1	100
B	5	6	4	20
C	1	2	0	60
D	3	4	2	80
Dummy	5	5	5	60
Demand (kgms)	120	140	60	

The initial solution is obtained by applying Vogel's approximation method.

Factory	Sales Centres			Capacity	Difference
	1	2	3		
A	100			100/0	11-
B		20		20/0	111
C			60	60/0	1-
D	20	60		80/60/-	111
Dummy		60		60/0	000
Demand	120/20/0	140/120/60/0	60/0		
differenc	1	1	1		
	1	1	-		
	2	1	-		

The solution obtained by VAM is as given below:

Factory	Sales Centre			Capacity
	1	2	3	
A	100			100
B		20		20
C			60	60
D	20	60		80
Dummy		60		60
Demand	120	140	60	



12.34 Advanced Management Accounting

The initial solution is tested for optimality. The total number of independent allocations is 6 which is one less than $(m + n - 1)$ allocations. Let us introduce a small quantity e in the least cost independent cell (A, 3). We also introduce $u_i, v_j, i = (1,2,3,4,5); j = (1,2,3)$ such that $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for allocation cells.

We assume $v_2 = 0$ and remaining v_i 's, v_j 's and Δ_{ij} are calculated as below:

Factory	Sales Centre			u_i
	1	2	3	
A	100	?	e	3
B	?	20	?	6
C	?	?	60	2
D	20	60	?	4
Dummy	1	60	2	5
V_j	-1	?	-2	

Since all $\Delta_{ij} \geq 0$ for the non allocated cells, hence the solution given by above matrix is optimal. The optimal solution for the given problem is given below:

From Factory	To sales Centre	Quantity	Profit per unit (Rs.)	Total profit (Rs.)
A	1	100	3	300
B	2	20	-1	-20
C	3	60	5	300
D	1	20	2	40
D	2	60	1	60
Dummy	2	60	0	0
			Total profit	<u>680</u>

(Note: Since some of the Δ_{ij} are equal to zero, alternate solution also exists).

Illustraton

Factory	Dstination			Supply to be exhausted
	1	2	3	
A	5	1	7	10
B	6	4	5	80
C	3	2	5	15
Demand	75	20	50	



The Transportation Problem 12.35

Since there is not enough supply, some of the demands at the three destinations may not be satisfied. For the unsatisfied demands, let the penalty costs be rupees 1,2 and 3 for destinations (1), (2) and (3) respectively. Find the optimal allocation that minimises the transportation and penalty cost.

Solution

Since demand ($= 75 + 20 + 50 = 145$) is greater than supply ($= 10 + 80 + 15 = 105$) by 40 units, the given problem is an unbalanced one. We introduce a dummy factory with a supply of 40 units. It is given that for the unsatisfied demands, the penalty cost is rupees 1,2 and 3 for destinations (1), (2) and (3) respectively. Hence, the transportation problem becomes

	Destination			Supply to be exhausted
	(1)	(2)	(3)	
A	5	1	7	10
B	6	4	6	85
C	3	2	5	10
Dummy	1	2	3	40
Demand	75	20	50	

The initial solution is obtained below by vogel's method"

Factory	Destination			Supply	Difference
	(1)	(2)	(3)		
A	5	10	7	10/0	4 --
B	20	10	50	80/70/50/0	222
C	15	2	5	15/0	111
Dummy	40	2	3	40/0	11--
Demand	75/35/20/0	20/10/0	50/0		
Defference	2	1	2		
	2	0	2		
	3	2	1		



12.36 Advanced Management Accounting

The initial solution is given in the table below:

Factory	Destination			Supply
	(1)	(2)	(3)	
A	2 5	10 1	7	10
B	20 6	10 4	50 6	80
C	15 3	2	5	15
Dummy	40 1	2	3	40
Demand	75	20	50	-5

We now apply the optimality test to find whether the initial solution found above is optimal or not.

The number of allocations is 6 which is equal to the required $m + n - 1 (=6)$ allocations. Also, these allocations are independent. Hence, both the conditions are satisfied.

Let us now introduce u_i and v_j ; $i = (1,2,3,4)$ and $j = (1,2,3)$ such that $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for allocated cells. We assume that $u_2 = 0$ and remaining u_i 's, v_j 's and Δ_j 's are calculated. Since all Δ_{ij} 's for non basic cells are positive, therefore, the solution obtained above is an optimal one. The allocation of factories to destinations and their cost is given below:-

Factor	Destination	units	Cost	Total cost	
A	2	10	Re. 1	Rs. 10	
B	1	20	Rs. 6	Rs. 120	Transportation cost
B	2	10	Rs. 4	Rs. 40	
B	3	10	Rs. 6	Rs. 40	Penalty cost
C	1	15	Rs. 3	Rs. 300	
Dummy	1	40	Re. 1	Rs. 40	
				<u>Rs. 555</u>	

Illustration

ABC enterprises is having three plants manufacturing dry-cells, located at different locations. Production cost differs from plant to plant. There are five sales offices of the company located in different regions of the country. The sales prices can differ from region to region. The shipping cost from each plant to each sales office and other data are given by following table:



The Transportation Problem 12.37

Production cost per unit	Production data table Maximum capacity in No. of units	Plant No.
20	150	1
22	200	2
18	125	3

Shipping cost & demand & sales price table

Shipping cost

	Sales office 1	Sales office 2	Sales office 3	Sales office 4	Sales office 5
Plant 1	1	1	5	9	4
Plant 2	9	7	8	3	6
Plant 3	4	5	3	2	7

Demand & sales prices

	Sales office 1	Sales office 2	Sales office 3	Sales office 4	Sales office 5
Demand	80	100	75	45	125
Sales price	30	32	31	34	29

Find the production and distribution schedule most profitable to the company

Solution

The given information can be tabulated in following transportation problem:

	Profit matrix Sales Offices					
Plant	1	2	3	4	5	Capacity in units
1	9	11	6	5	5	150
2	-1	3	1	9	1	200
3	8	9	10	14	4	125
Demand	80	100	75	45	125	

Where entries in the cells of the above table indicate profit per unit received by selling one unit of item from plant i ($i=1,2,3$) to the sales office ($i=1,2,3,4,5$). The profit per unit is calculated using the following formula.

$$\text{Profit} = \text{Sales price} - (\text{Production cost} + \text{Shipping cost})$$

The objective of the company is to maximize the profit. For achieving this objective, let us convert this maximization problem into minimization problem by subtracting all the elements of the above payoff matrix from the highest payoff i.e. Rs. 14.



12.38 Advanced Management Accounting

Plant	Loss Matrix Sales Offices					Capacity in units
	1	2	3	4	5	
1	5	3	8	9	9	150
2	15	11	13	5	13	200
3	6	5	4	0	10	125
Demand	80	100	75	45	125	

The problem is an unbalanced transportation problem since capacity (=475 units) is 50 units more than the demand. Hence a dummy sales office is added with cost equal to zero for all plants and demand equal to 50 units. Now, let us apply Vogel's Approximation method to the resultant balanced matrix for finding the initial feasible solution.

Plant	1	2	3	4	5	Dummy	Capacity	Difference
1	50	100					150/50/0	3/3/2/2/4
2	25				125	50	200/150/125/0	5/11/2/2/2/2
3	5		75	45			125/80/5/0	0/4/1/1/4/4/
Demand	80/30/ 25/0	100/0	75/0	45/0	125/0	50/0		
Diff.	1	2	4	5	1	0		
	1	2	4	-	1	0		
	1	2	4	-	1	-		
	1	2	-	-	1	-		
	1	-	-	-	1	-		

The initial solution obtained by VAM is given below which is tested for optimality.



The Transportation Problem 12.39

Plant	1	2	3	4	5	Dummy	Capacity in units
1	50	100					150
	5	3	8	9	9	0	
2	25				125	50	200
	15	11	13	5	13	0	
3	5		75	45			125
	6	5	4	0	10	6	
Demand in units	80	100	75	45	125	50	

These are $m + n - 1 = 8$ independent allocations. Let us now introduce $u_i, v_j, i = (1,2,3); j = (1,2, \dots, 6)$ such that $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for allocation cells. We assume $u_2 = 0$ and remaining u_i 's v_j 's and Δ_{ij} 's are calculated as below:

Sales Offices							
Plant	1	2	3	4	5	Dummy	u_i 's
1	50	100	5	10	6	10	
	5	3	8	9	9	0	-10
2	25	?	-2	10	-4	?	
	15	11	13	5	13	0	0
3	5	0	1	75	45	-?	
	6	5	4	0	10	9	-9
V_j 's	15	13	13	9	13	0	

Since some of the Δ_{ij} 's are negative, therefore, the above solution is not optimal. Introduce in the cell (2,4) with the most negative Δ_{ij} , an assignment. The value of θ and reallocated solution as obtained from above is given below. The reallocated solution is again tested for optimality. Hence, the values u_i 's v_j 's and Δ_{ij} 's are again calculated.

Sales Offices							
Plant	1	2	3	4	5	Dummy	u_i 's
1	50	100	5	10	2	6	
	5	3	8	9	9	0	-6
2	4	2	4	25	125	50	
	15	11	13	5	13	0	0
3	30	1	75	20	2	5	
	6	5	4	0	10	0	-5
V_j 's	11	9	9	5	13	0	



12.40 Advanced Management Accounting

Since all Δ_{ij} 's for non-basic cells are positive, therefore, the solution obtained above is an optimal one. The allocation of plants to sales officers and their profit amount is given below:

Plant	Sales Office	units	profit per	profit unit
1	1	50	9	450
1	2	100	11	1,100
2	4	25	9	225
2	5	125	1	125
2	Dummy	50	0	0
3	1	30	8	240
3	3	75	10	750
3	4	20	14	<u>280</u>
Total				<u>3,170</u>

SUMMARY

- Finding of initial base feasible solution
 - ✓ North west corner method
 - ✓ Least cost method
 - ✓ Vogels approximation method
- Optimality test can be applied if it satisfies the given two conditions:
 - ✓ It contains exactly $m+n-1$ allocations where m and n represent the number of rows and column of the table.
 - ✓ These allocations are independent i.e. a loop can not be performed by them.

SELF EXAMINATION QUESTIONS

1. Find the optimal solution of the following transportation problem :

	D_1	D_2	D_3	D_4	Available
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Required	22	35	25	41	123

The cell entries are unit transportation costs.

(Answer : Opt. $z = 2221$).



The Transportation Problem 12.41

2. Solve the following transportation problem :

	D ₁	D ₂	D ₃	D ₄	D ₅	Available
O ₁	9	12	9	6	9	5
O ₂	7	3	7	7	5	4
O ₃	6	5	9	11	3	2
O ₄	6	8	11	2	2	9
Required	4	4	6	2	4	20

(Answer : Opt. $z = 102$).

3. Solve the following transportation problem :

				Available
	8	7	3	60
	3	8	9	70
	11	3	5	80
Required	50	80	80	

(Answer : Opt. $z = 75$).

4. Solve the following transportation problem :

		Stores						
		1	2	3	4	5	6	
Factories	1	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
Availability		4	4	6	2	4	2	